arXiv:astro-ph/9510134v1 25 Oct 1995

A LINEAR PROGRAMMING APPROACH TO INHOMOGENEOUS PRIMORDIAL NUCLEOSYNTHESIS

Richard E. Leonard and Robert J. Scherrer
Department of Physics
The Ohio State University
Columbus, OH 43210

ABSTRACT

We examine inhomogeneous primordial nucleosynthesis for *arbitrary* distributions f of the baryon-to-photon ratio η, in the limit where neither particle diffusion nor gravitational collapse is important. By discretizing $f(\eta)$ and using linear programming, we show that for a set of m constraints on the primordial element abundances, the maximum and minimum possible values of $\bar{\eta}$ (the final mean value of η) are given when $f(\eta)$ is a sum of at most $m + 1$ delta functions. Our linear programming results indicate that when f is taken to be an arbitrary function, there is no lower bound on $\bar{\eta}$, while the upper bound is essentially the homogeneous upper bound.

1. INTRODUCTION

One of the most widely-investigated variations in primordial nucleosynthesis is the possibility that the baryon density is inhomogeneous at the time of nucleosynthesis. In the simplest such inhomogeneous models, nucleosynthesis is assumed to take place independently in regions of different density, with the final element abundances derived by simply mixing the matter from these different regions (Epstein & Petrosian 1975; Barrow & Morgan 1983; Yang et al. 1984; Copi, Olive, & Schramm 1995, hereafter COS). Such a treatment is valid if the dominant fluctuations are of sufficiently low amplitude and on a sufficiently large length scale that neither gravitational collapse nor particle diffusion is important. If the length-scale of the fluctuations is sufficiently short, then particle diffusion before and during nucleosynthesis can significantly alter the element yields (Applegate, Hogan, & Scherrer 1987; Alcock, Fuller, & Mathews 1987; for more recent studies, see Thomas, et al. 1994; Jedamzik, Fuller, & Mathews 1994). On the other hand, if the fluctuation amplitudes are high enough, the highest density regions can undergo gravitational collapse, effectively removing the elements in those regions from the final measured abundances. This scenario has also been extensively explored (Sale & Mathews 1986; Gnedin, Ostriker, & Rees 1995; COS; Jedamzik & Fuller 1995).

Here we consider only the simplest of these scenarios (no diffusion or collapse), but we provide a significant generalization to previous work. All previous studies have assumed a particular form for the distribution of baryon-to-photon ratio $f(\eta)$. In this paper, we treat the case of *arbitrary* $f(\eta)$. Obviously, a brute-force calculation of this case, examining all possible functions, would be impossible. Even the analysis of a limited subset of the infinite number of possibilities is an enormous computational task. However, by discretizing the problem, we can turn it into a problem in linear programming with a well-defined set of easily-obtained solutions. In the next section, we explain our linear programming method and calculate the density distributions which give the minimum and maximum possible values for the mean value of η for a reasonable set of observational constraints on the primordial element abundances. Our conclusions are discussed in §3. We find that there is no lower bound on η (consistent with the earlier arguments of Jedamzik, Mathews, & Fuller 1995), while the upper bound is essentially the homogeneous upper bound.

2. CALCULATIONS

The most recent and complete treatment of inhomogeneous nucleosynthesis when post-nucleosynthesis mixing is the only important effect has been given by COS, so we follow their treatment. The elements produced in detectable quantities in primordial nucleosynthesis are ^{4}He, D, ^{3}He, and ^{7}Li. The observational limits on the abundances of these elements have been discussed in great detail in a number of recent papers (Walker et al. 1991; Copi, Schramm, & Turner 1995a,b; Hata, et al. 1995, 1996). However, since we wish to compare

our results to the calculations of COS, we will use the abundance limits given there:

$$
\begin{aligned}
0.221 &\leq Y_P \leq 0.243, \\
(D/H) &\geq 1.8 \times 10^{-5}, \\
(D +^3 He)/H &\leq 1.0 \times 10^{-4}.
\end{aligned}
\tag{1}
$$

COS used two different bounds on the ^{7}Li abundance:

$$
\begin{aligned}
(^7Li/H) &\leq 1.4 \times 10^{-10}, \\
(^7Li/H) &\leq 2.0 \times 10^{-10},
\end{aligned}
\tag{2}
$$

Although we will use these limits in all of our calculations, our method of calculation, which is the main new idea in this paper, can be applied to any set of limits.

Now assume that the distribution of densities is given by an unknown function $f(\eta)$. Previous studies have assumed a variety of functions for $f(\eta)$, including the gamma distribution (Epstein & Petrosian 1975; Yang et al. 1984) and the lognormal distribution (Barrow & Morgan 1983). COS considered both of these as well as a Gaussian distribution for η. We consider here the case of arbitrary $f(\eta)$. The mean final value for the baryon-to-photon ratio, $\bar{\eta}$, is given by

$$
\bar{\eta} = \int_0^\infty f(\eta) \, \eta \, d\eta
\tag{3}
$$

The final element abundances are mass-weighted averages of the element abundances produced in the individual regions. If $\bar{X}_A$ is the mean mass fraction of nuclide A measured today, then

$$
\bar{X}_A = \int_0^\infty X_A(\eta) \, f(\eta) \, \eta \, d\eta/\bar{\eta},
\tag{4}
$$

where $X_A(\eta)$ is the mass fraction of nuclide A produced in standard (homogeneous) nucleosynthesis with baryon-to-photon ratio η. Since f represents a probability distribution, it must be normalized:

$$
\int_0^\infty f(\eta) \, d\eta = 1.
\tag{5}
$$

Suppose that we wanted to test all possible functions f. One way to do this would be to divide the range in η into discrete bins, and to approximate the integrals in equations (3) and (4) as sums:

$$
\bar{\eta} = \sum_i f_i \eta_i \Delta \eta_i
\tag{6}
$$

and

$$
\bar{X}_A = \sum_i X_{Ai} f_i \eta_i \Delta \eta_i / \bar{\eta}
\tag{7}
$$

where the η dependence of f and X_A is expressed through their dependence on the bin number i. The bins need not all be of equal size, hence the factor $\Delta \eta_i$. The normalization constraint (equation 5) becomes

$$
\sum_i f_i \Delta \eta_i = 1.
\tag{8}
$$

Now one could imagine doing a Monte Carlo simulation, scanning through all possible distributions f_i which satisfy the constraint in equation (8). In practice, this is impossible. For example, if we divided the range in η into 100 bins, and divided f_i into 1000 "units" of magnitude 0.001 to be distributed among these bins, we would have to calculate $C(1099, 1000) \sim 10^{144}$ sets of element abundances, a calculation clearly beyond all but the most robust graduate students. However, most discussions of inhomogeneous nucleosynthesis center on a much simpler problem: given a set of constraints such as those given in equations (1)-(2), and a function or family of functions f for the distribution of η, what are the largest and smallest allowed values for $\bar{\eta}$? If we express the problem this way, discretizing it as in equations (6) and (7), then the question of maximizing or minimizing the value of η is reduced to a problem in linear programming. (A discussion of both the theory and practice of linear programming can be found in, for example, Press, Flannery, Teukolsky, & Vetterling 1992, from which some of the following discussion is taken).

The fundamental problem in linear programming is the following: given a set of N non-negative independent variables x_j, and a set of M constraints of the form:

$$\sum_{j=1}^{N} a_j x_j \leq b, \tag{9}$$

or

$$\sum_{j=1}^{N} a_j x_j \geq b, \tag{10}$$

or

$$\sum_{j=1}^{N} a_j x_j = b, \tag{11}$$

maximize or minimize the function

$$z = \sum_{j=1}^{N} c_j x_j \tag{12}$$

The linear programming nature of our discretized problem above becomes clearer if we define the quantities p_i to be

$$p_i \equiv f_i \Delta \eta_i. \tag{13}$$

Then equations (6)-(8) become

$$\bar{\eta} = \sum_i p_i \eta_i, \tag{14}$$

$$\bar{X}_A = \sum_i X_{Ai} p_i \eta_i / \bar{\eta}, \tag{15}$$

and

$$\sum_i p_i = 1. \tag{16}$$

Equations (14) - (16) are now clearly in the form of a linear programming problem: the N independent variables are the p_i's, equations (15) and (16) provide the constraint equations,

and $\bar{\eta}$ given by equation (14) is the quantity we seek to maximize or minimize. Note that equation (15) contains $\bar{\eta}$ in the denominator, so that the bounds $\bar{X}_A < X_{\text{upperbound}}$ and $\bar{X}_A > X_{\text{lowerbound}}$ do not immediately translate into linear programming constraints of the form given by equations (9)-(11). However, using equations (14) and (15), we can rewrite these bounds as

$$\sum_i [X_{Ai} - X_{\text{upperbound}}]p_i\eta_i \leq 0, \tag{17}$$

and

$$\sum_i [X_{Ai} - X_{\text{lowerbound}}]p_i\eta_i \geq 0, \tag{18}$$

which are in the form of equations (9) and (10).

An additional complication is the fact that our constraints on all of the elements other than ^{4}He are expressed in terms of number ratios to hydrogen, $(A/H) \equiv n_A/n_H$, rather than as mass fractions, X_A, while our prescription for mixing the various element abundances uses the mass fractions. However, it is easy to translate the number ratio bounds into a suitable form. Recall that

$$(A/H) = \frac{X_A}{A X_H}, \tag{19}$$

where X_H is the ^{1}H mass fraction. Then an observational upper bound on (A/H) of the form $(A/H) \leq (A/H)_{\text{upperbound}}$ can be written in the form

$$\frac{\bar{X}_A}{A\bar{X}_H} \leq (A/H)_{upperbound} \tag{20}$$

where both $\bar{X}_A$ and $\bar{X}_H$ are given by equation (15). Substituting for $\bar{X}_A$ and $\bar{X}_H$ from equation (15), we obtain

$$\sum_i [X_{Ai} - (A/H)_{\text{upperbound}}A X_{Hi}]p_i\eta_i \leq 0. \tag{21}$$

Note that the expression for $\bar{\eta}$ has dropped out of the equation, but now the inequality includes a sum over X_{Hi}. For the lower bound $(A/H) \geq (A/H)_{\text{lowerbound}}$ we obtain a similar result:

$$\sum_i [X_{Ai} - (A/H)_{\text{lowerbound}}A X_{Hi}]p_i\eta_i \geq 0. \tag{22}$$

Equations (21) and (22) are both in the form of acceptable linear programming constraints.

We now note an important result of linear programming theory: given a set of N variables and M constraints, the solution which maximizes or minimizes z has at least $N - M$ of the variables x_j equal to zero (Press et al. 1992). In our case, if we have m constraints on the element abundances (e.g., equations (1) and (2) give $m = 5$ constraints), and the normalization condition (equation 8) provides one additional constraint, then at most $M = m+1$ of the p_i's are non-zero; in this case $M = 6$. If we take the continuum limit of equations (6)-(8), we arrive at the central result of this paper: for a set of m constraints on the element

abundances such as those given in equations (1) and (2), and an arbitrary distribution of η given by the function $f(\eta)$, the largest and smallest possible values for $\bar{\eta}$ occur when $f(\eta)$ is the sum of at most $m + 1$ delta functions.

We use the simplex method (Press et al. 1992) to solve our discretized problem. We ran the primordial nucleosynthesis code of Wagoner (Wagoner, Fowler, & Hoyle 1967) as updated by Kawano (1992), with a neutron lifetime of $\tau = 887$ sec. [This differs slightly from the neutron lifetime in COS, and we do not include the small correction factor to the ^{4}He production used by COS, but these are small differences which will not significantly affect our results]. We allowed η to vary from 10^{-13} to 10^{-7}, and we divided this interval into 600 equally-spaced logarithmic bins, calculating the mass fractions of ^{4}He, ^{3}He, D, and ^{7}Li for all 600 values of η. We also included a bin corresponding to zero baryon density and zero element production. We then use the simplex method to determine the form for p_i which maximimizes or minimizes $\bar{\eta}$ for the abundance constraints given in equations (1) and (2).

When we attempt to minimize $\bar{\eta}$, we find that almost all of p_i is concentrated in the lowest bin (i.e., the bin corresponding to zero baryon density). In fact, there is no "lowest value" for $\bar{\eta}$; the mean baryon-to-photon ratio can be arbitrarily small, a point recently emphasized by Jedamzik, Mathews, and Fuller (1995). For example, one could take $f(\eta) = p_1\delta(\eta - \eta_0) + p_2\delta(\eta)$, where η_0 is a value for η which gives acceptable nucleosynthesis for the homogeneous case. By mixing the correct homogeneous η with the baryon-free regions, we obtain the correct element abundances regardless of the values of p_1 and p_2, but by taking the limit $p_1 \to 0$, $p_2 \to 1$, the value for $\bar{\eta}$ can be made arbitrarily small.

A more interesting question is the upper bound on $\bar{\eta}$. Using either lithium bound in equation (2), we find only two non-zero bins, straddling the upper bound on η in the homogeneous model. To resolve this function further, we re-ran the code using 1000 bins between $\eta = 10^{-10}$ and $\eta = 10^{-9}$. Using the first lithium bound in equation (2) we obtain:

$$f(\eta) = (0.21)\delta(\eta - 3.31 \times 10^{-10}) + (0.79)\delta(\eta - 3.33 \times 10^{-10}), \qquad (23)$$

which gives $\bar{\eta} = 3.33 \times 10^{-10}$. In this case, it is the limit on ^{7}Li which is saturated. The second lithium bound in equation (2) yields the solution:

$$f(\eta) = (0.58)\delta(\eta - 3.40 \times 10^{-10}) + (0.42)\delta(\eta - 3.44 \times 10^{-10}), \qquad (24)$$

with $\bar{\eta} = 3.42 \times 10^{-10}$. For this case, the ^{4}He limit is saturated. In fact, given the precision with which we calculate the various element abundances, the difference between these bounds and the homogeneous upper bound is not significant. We are led to conclude that the homogeneous upper bound on η cannot be exceeded for *any* distribution $f(\eta)$. Our results are consistent with the claim that $f(\eta)$ should be at most a sum of $m + 1$ delta functions; in this case $m + 1 = 6$, while our optimum solution is the sum of only two delta functions.

Another interesting question is whether the inclusion of inhomogeneities can resolve a slight discrepancy between the predictions of primordial nucleosynthesis and the observations.

A number of recent studies (Copi, Schram, & Turner 1995a,b; Hata, et al. 1995) have suggested that values of η which give production of D, ^{3}He, and ^{7}Li consistent with the observations will overproduce ^{4}He, so it has been suggested, for example, that the "true" primordial ^{4}He abundance is somewhat larger than is currently reported by the observers. Since inhomogeneities allow for the reduction of $\bar{\eta}$ by an arbitrary amount, is it tempting to think that a reduction in the ^{4}He abundance might also be possible. To test this, we have used the D, ^{3}He, and ^{7}Li limits given above, while minimizing the value of ^{4}He. Our linear programming method cannot be applied directly to minimize ^{4}He, because the expression for ^{4}He contains a factor proportional to $\bar{\eta}$. However, we can find the distribution in p_i which minimizes ^{4}He for any given value of $\bar{\eta}$, so we have simply scanned over a range in $\bar{\eta}$ to find the smallest ^{4}He as a function of $\bar{\eta}$. Using this method, we find that no significant reduction in ^{4}He is possible. Again, our results apply to arbitrary distributions of η.

3. DISCUSSION

We find that for this simplest model for inhomogeneous nucleosynthesis, there is no lower bound on η, as expected (Jedamzik, Mathews, & Fuller 1995). Previous studies which assumed particular functional forms for $f(\eta)$ all produced a fairly narrow range in the allowed values for $\bar{\eta}$ (Epstein & Petrosian 1975; Barrow & Morgan 1983; Yang et al. 1984; COS). However, this occurred because all of the functions $f(\eta)$ in these papers were unimodal, i.e., characterized by a single maximum. Hence, they cannot approximate the sort of solution which has two large peaks at $\eta = 0$ and $\eta = \eta_0$ (where η_0 gives acceptable abundances in the homogeneous model). Our upper bound on $\bar{\eta}$ is, for all practical purposes, the homogeneous upper bound on η. This is consistent with the results of COS; for all three functions they examined, it is clear from their figures that the upper bound on $\bar{\eta}$ is no larger than that obtained when the variance of $f(\eta)$ goes to zero. The importance of our results is that they give the most general upper and lower bounds on $\bar{\eta}$ for *any* density distribution, essentially bringing to a close the two-decade-long investigations of these simplest inhomogeneous models. Our final conclusion is that for the inhomogeneous models in which mixing is the only important process, $\bar{\eta}$ can be arbitrarily small (a point already noted by Jedamzik, Mathews, & Fuller 1995), but $\bar{\eta}$ cannot be larger than the homogeneous upper bound on η.

Linear programming is not a technique usually applied to astrophysical problems, although it has been previously used in galactic dynamics (Schwarzschild 1979). Our technique of discretizing the problem and using linear programming could be applied to any problem with constraints on integrals of an unknown function. This technique cannot be applied to inhomogeneous nucleosynthesis when particle diffusion is significant. However, it could be applied to the case when collapse of high density regions is important. We have chosen not to address this case because it involves many particular model assumptions, but, for example, equation (21) of COS can easily be put into the form of a linear programming constraint.

REFERENCES

Alcock, C.R., Fuller, G.M., & Mathews, G.J. 1987, ApJ, 320, 439

Applegate, J.H., Hogan, C.J., & Scherrer, R.J. 1987, Phys Rev D, 35, 1151

Barrow, J.D., & Morgan, J. 1983, MNRAS, 203, 393

Copi, C.J., Olive, K.A., & Schramm, D.N. 1995, ApJ, 451, 51 (COS)

Copi, C.J., Schramm, D.N., & Turner, M.S., 1995a, Science, 267, 192

Copi, C.J., Schramm, D.N., & Turner, M.S., 1995b, Phys Rev Lett, submitted

Epstein, R.I., & Petrosian, V. 1975, ApJ, 197, 281

Gnedin, N.Y., Ostriker, J.P., & Rees, M.J. 1995, ApJ, 438, 40

Hata, N., Scherrer, R.J., Steigman, G., Thomas, D., & Walker, T.P. 1996, ApJ, in press

Hata, N., et al. 1995, Phys Rev Lett, in press

Jedamzik, K., & Fuller, G.M. 1995, ApJ, 452, 33

Jedamzik, K., Fuller, G.M., & Mathews, G.J. 1994, ApJ, 423, 50

Jedamzik, K., Mathews, G.J., & Fuller, G.M. 1995, ApJ, 441, 465

Kawano, L., 1992, FERMILAB-Pub-92/04-A

Press, W.H., Flannery, B.P., Teukolsky, S.A., & Vetterling, W.T. 1992, Numerical Recipes, 2nd edition, (Cambridge: Cambridge University Press)

Sale, K.E., & Mathews, G.J. 1986, ApJ, 309, L1

Schwarzschild, M. 1979, ApJ, 232, 236

Thomas, D., et al. 1994, ApJ, 430, 291

Wagoner, R.V., Fowler, W.A., and Hoyle, F. 1967, ApJ, 148, 3

Walker, T.P., Steigman, G., Schramm, D.N., Olive, K.A., & Kang, H. 1991, ApJ, 376, 51

Yang, J., Turner, M.S., Steigman, G., Schramm, D.N., & Olive, K.A. 1984, ApJ 281, 493